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POLYTOPE SUMS AND LIE CHARACTERS

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This paper is dedicated to the late Professor R. T. Sharp.

Abstract. A new application of polytope theory to Lie theory is presented. Exponential sums of convex lattice polytopes are applied to the characters of irreducible representations of simple Lie algebras. The Brion formula is used to write a polytope expansion of a Lie character, that makes more transparent certain degeneracies of weight-multiplicities beyond those explained by Weyl symmetry.

1. Introduction

A polytope is the convex hull of finitely many points in $\mathbb{R}^d$ (see [15], for example). A lattice polytope (also known as an integral polytope) is a polytope with all its vertices in an integral lattice $\Lambda \in \mathbb{R}^d$ (see [3], for example). The exponential sum of a lattice polytope Pt,

$$\sum_{x\in Pt\cap\Lambda} \exp\{\langle c,x\rangle\}\ , \tag{1.1}$$

is a useful tool. Here c is a vector in $\mathbb{R}^d$, and $\langle\cdot,\cdot\rangle$ is the usual inner product. For simplicity of notation, we'll consider *formal* exponential sums

$$E[Pt;\Lambda]\ :=\ \sum_{x\in Pt\cap\Lambda} e^x\ . \tag{1.2}$$

Here the formal exponential e^x satisfies

$$e^x e^y\ =\ e^{x+y}\ , \tag{1.3}$$

for all $x, y \in \Lambda$, and simply stands for the function $e^x(c) := e^{\langle c,x\rangle}$. These exponential sums are the generating functions for the integral points in a lattice polytope.

As an extremely simple example, consider the one-dimensional lattice polytope with vertices (2) and (7). Its exponential sum is

$$e^{(7)} + e^{(6)} + e^{(5)} + e^{(4)} + e^{(3)} + e^{(2)}\ . \tag{1.4}$$

As a very simple example, consider the lattice polytope in $\mathbb{Z}^2$ with vertices $(0,0), (1,0)$ and $(1,1)$. Its exponential sum is just

$$e^{(1,1)} + e^{(1,0)} + e^{(0,0)}\ , \tag{1.5}$$

since its vertices are the only lattice points it contains.

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We will apply knowledge of these polytope sums to the calculation of the characters of representations of simple Lie algebras (Lie characters, for short). Polytope theory and Lie theory may have much to teach each other, and we hope this contribution will prompt others.

This work is preliminary. Proofs are not given, and we concentrate on low-rank examples. A fuller treatment will be given in [8].

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2. Brion formula

Brion [5] has proved a formula for the exponential sum of a convex lattice polytope, that expresses it as a sum of simpler terms, associated with each of the vertices of the polytope. It reads

$$E[Pt;\Lambda] \;=\; \sum_{x\in Pt\cap\Lambda} e^x \;=\; \sum_{v\in \mathrm{Vert}Pt} e^v\,\sigma_v\ . \tag{2.1}$$

Here $\mathrm{Vert}Pt$ is the set of vertices of Pt.

The vertex term σ_v is determined by the cone K_v, defined by extending the vertex $v\in\mathrm{Vert}Pt$,

$$K_v \;=\; \{\, x\,|\, v+\epsilon x\in Pt,\ \text{for all sufficiently small } \epsilon>0\,\}\ . \tag{2.2}$$

In other words, K_v is generated by the vectors $u_i=v_i-v$, where $[v_i,v]$ is an edge of Pt, that indicate the feasible directions at v; it is the cone of feasible directions at v.[1]

σ_v is associated with the exponential sum of K_v. Precisely, if K_v is generated by vectors $u_1,\ldots,u_k\in\Lambda$, then the series $\sum_{x\in K_v\cap\Lambda}\exp\{\langle c,x\rangle\}$ converges for any c such that $\langle c,u_i\rangle<0$, for all $i=1,\ldots,k$. The series defines a meromorphic function of c, $\sigma_v(c)$ [3].

A cone K_v is unimodular if the fundamental parallelopiped bounded by its feasible directions u_i contains no lattice points in its interior. If K_v is unimodular, its exponential sum takes the very simple form of a multiple geometric series

$$\sigma_v \;=\; \prod_{i=1}^{d}\,(\,1-e^{u_i}\,)^{-1}\ . \tag{2.3}$$

Pt is totally unimodular if K_v is unimodular for all $v\in\mathrm{Vert}Pt$. If a cone is not unimodular, its exponential sum is not so simple, but is still easy to write; the fundamental parallelopiped of a non-unimodular cone is always a finite union of unimodular parallelopipeds.

For the extremely simple one-dimensional example considered above, the Brion formula gives

$$\frac{e^{(7)}}{1-e^{(-1)}} \;+\; \frac{e^{(2)}}{1-e^{(1)}}\ . \tag{2.4}$$

[1] We will assume that the vectors u_i are linearly independent for all $v\in\mathrm{Vert}Pt$, i.e., that all the K_v are simple cones, and so Pt is a simple polytope.

If we use

$$\frac{1}{1-e^x} \;=\; \frac{-e^{-x}}{1-e^{-x}}\ , \tag{2.5}$$

this becomes

$$\frac{e^{(7)}-e^{(1)}}{1-e^{-(1)}}\ , \tag{2.6}$$

i.e., the simple geometric series, so that (1.4) is recovered.

For the $d=2$ example above, the Brion formula (2.1),(2.3) gives

$$\begin{aligned}&\frac{e^{(1,1)}}{(1-e^{(-1,-1)})(1-e^{(0,-1)})} \;+\; \frac{e^{(1,0)}}{(1-e^{(-1,0)})(1-e^{(0,1)})}\\ &\qquad\qquad +\; \frac{e^{(0,0)}}{(1-e^{(1,0)})(1-e^{(1,1)})}\ .\end{aligned} \tag{2.7}$$

This simplifies to (1.5).

3. Weyl character formula

Let $P=\mathbb{Z}\{\Lambda^i\,|\,i=1,\ldots,r\}$ denote the weight lattice of a simple Lie algebra X_r, of rank r. Here Λ^i stands for the i-th fundamental weight. $R_>$ ($R_<$) will denote the set of positive (negative) roots of X_r, and $S=\{\alpha_i\,|\,i=1,\ldots,r\}$ its simple roots.

The highest weights of integrable irreducible representations of X_r belong to the set $P_\geq=\{\lambda=\sum_{j=1}^r\lambda_j\Lambda^j\,|\,\lambda_j\in\mathbb{Z}_{\geq 0}\}$. The formal character of the irreducible representation $L(\lambda)$ of highest weight λ is

$$\mathrm{ch}_\lambda \;=\; \sum_{\mu\in P}\mathrm{mult}_\lambda(\mu)\, e^\mu\ , \tag{3.1}$$

where $\mathrm{mult}_\lambda(\mu)$ denotes the multiplicity of weight μ in $L(\lambda)$. The famous Weyl character formula is

$$\mathrm{ch}_\lambda \;=\; \frac{\sum_{w\in W}(\det w)\, e^{w.\lambda}}{\prod_{\alpha\in R_>}(1-e^{-\alpha})}\ . \tag{3.2}$$

Here $W=\langle r_i\,|\,i=1,\ldots,r\rangle$ is the Weyl group of X_r, and r_i is the i-th primitive Weyl reflection, with action $r_i\mu=\mu-\mu_i\alpha_i$ on $\mu\in P$. $\det w$ is the sign of $w\in W$, and $w.\lambda=w(\lambda+\rho)-\rho$ is the shifted action of w on λ, with the Weyl vector $\rho=\sum_{i=1}^r\Lambda^i=\sum_{\alpha\in R_>}\alpha/2$.

After using $\mathrm{ch}_0=1$ to derive the Weyl denominator formula, we can rewrite the character formula as

$$\mathrm{ch}_\lambda \;=\; \frac{\sum_{w\in W}(\det w)\, e^{w(\lambda+\rho)}}{\sum_{w\in W}(\det w)\, e^{w\rho}}\ . \tag{3.3}$$

Comparing to (3.1) reveals the Weyl symmetry

$$\mathrm{mult}_\lambda(\mu) \;=\; \mathrm{mult}_\lambda(w\mu) \qquad (\forall w\in W) \tag{3.4}$$

of the weight multiplicities.

Alternatively, we can rewrite (3.2) in a different form that is also manifestly W symmetric:

$$\mathrm{ch}_\lambda \ = \ \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in R_>} (1-e^{-w\alpha})^{-1} \ . \tag{3.5}$$

The usual formula (3.2) is recovered from this using (2.5). Each Weyl element $w\in W$ separates the positive roots into two disjoint sets:

$$\begin{array}{c} R^w_> := \{\alpha\in R_> \,|\, w\alpha\in R_>\} \ , \ \ R^w_< := \{\alpha\in R_> \,|\, w\alpha\in R_<\} \ , \\ R^w_>\cup R^w_< \ = \ R_> \ , \ \ R^w_>\cap R^w_< \ = \ \{\} \ , \\ wR^w_> \ = \ R^w_> \quad , \quad wR^w_< \ = \ -R^w_< \ . \end{array} \tag{3.6}$$

It can be shown that $\det w = (-1)^{\|R^w_<\|}$, and

$$-w\rho+\rho \ = \ \sum_{\beta\in R^w_<} \beta \ . \tag{3.7}$$

Using these results, (3.5) becomes

$$\begin{aligned} \mathrm{ch}_\lambda \ &= \ \sum_{w\in W} e^{w\lambda} \prod_{\beta\in R^w_<} (-e^{w\beta})(1-e^{w\beta})^{-1} \prod_{\alpha\in R^w_>} (1-e^{-w\alpha})^{-1} \\ &= \ \sum_{w\in W} (\det w)\, e^{w\lambda - w\sum_{\gamma\in R^w_<}\gamma} \prod_{\beta\in R^w_<} (1-e^{w\beta})^{-1} \prod_{\alpha\in R^w_>} (1-e^{-w\alpha})^{-1} \ , \end{aligned} \tag{3.8}$$

so that (3.2) follows.

4. Character polytope-expansion

The form (3.5) of the Weyl character formula is similar to the Brion formula. To make this more precise, we'll write the Brion formula for the exponential sum of the lattice polytope Pt_λ with vertices in the Weyl orbit of a highest weight $\lambda\in P_\geq$, i.e. $\mathrm{Vert}Pt_\lambda = W\lambda$. The appropriate lattice here is the root lattice Q of X_r, shifted by λ: $\lambda+Q\subset P$. If we define

$$P(\lambda) \ := \ \{\, \mu \,|\, \mathrm{mult}_\lambda(\mu) \neq \ 0 \ \} \ \subset \ P \, , \tag{4.1}$$

then

$$Pt_\lambda \cap (\lambda+Q) \ = \ P(\lambda) \ , \tag{4.2}$$

and the polytope sum will be

$$E[Pt_\lambda;\lambda+Q] \ = \ \sum_{\mu\in P(\lambda)} e^\mu \ . \tag{4.3}$$

Let us call the so-defined polytope Pt_λ the *weight polytope*.

If the weight λ is regular, and the weight polytope is totally unimodular, then it is easy to see that the Brion formula gives

$$B_\lambda \ = \ \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in S} (1-e^{-w\alpha})^{-1} \ , \tag{4.4}$$

since the feasible directions at the vertex $w\lambda$ are just the Weyl-transformed simple roots $\{w\alpha \,|\, \alpha\in S\}$.

The last formula is remarkably similar to (3.5). We propose here to exploit this similarity, by expanding

$$\mathrm{ch}_\lambda \;=\; \sum_{\mu\le\lambda} A_{\lambda,\mu}\, B_\mu \ . \tag{4.5}$$

Here $\mu \le \lambda$ means $\lambda-\mu \in \mathbb{Z}_{\ge 0}R_>$, as usual. The constraint on the sum implies that the $A_{\lambda,\mu}$ are the entries of a triangular matrix. The character polytope-expansion (4.5) will manifest weight-multiplicity degeneracy beyond Weyl symmetry.

Before considering examples, let us mention two possible complications. First, the weight λ may not be regular, so that some subgroup of the Weyl group W stabilizes it. We still find that the exponential sum for Pt_λ equals B_λ, as written in (4.4). Also, when the algebra is not simply-laced, and λ is not regular, then the corresponding weight polytope Pt_λ may not be totally unimodular. Remarkably, even in that case, (4.4) seems to be the appropriate formula.

Consider a simple example, the adjoint representation of G_2. If α_2 is the short simple root, then the highest weight λ of the adjoint representation is $\Lambda^1 = \theta$. (Here θ will be used to denote the highest root.) The feasible directions at the vertex $\lambda = \Lambda^1$ are $-\alpha_1$ and $-\alpha_1 - 3\alpha_2$, *not* $-\alpha_1$ and $-\alpha_2$. The cone is not unimodular. Using (2.3), the cone function would be

$$(1-e^{-\alpha_1})^{-1}(1-e^{-\alpha_1-3\alpha_2})^{-1} \ . \tag{4.6}$$

This cone function would miss points – in this example, the consequent formula for the polytope Pt_λ would not have contributions from the short roots.

As pointed out in [3], a cone can be decomposed into a set of unimodular ones, and (2.3) can then be applied. In general, however, it is more efficient to use a signed decomposition into unimodular cones. In this example, drawing a G_2 root diagram shows that

$$\begin{aligned} e^\lambda\sigma_\lambda \;&= e^{\Lambda^1}(1-e^{-\alpha_1})^{-1}(1-e^{-\alpha_2})^{-1} \\ &\quad -\; e^{\Lambda^1-\alpha_2}(1-e^{-\alpha_1-3\alpha_2})^{-1}(1-e^{-\alpha_2})^{-1} \ . \end{aligned} \tag{4.7}$$

But this is just

$$\begin{aligned} e^\lambda\sigma_\lambda \;&= e^{\Lambda^1}(1-e^{-\alpha_1})^{-1}(1-e^{-\alpha_2})^{-1} \\ &\quad +\; e^{r_2\Lambda^1}(1-e^{-r_2\alpha_1})^{-1}(-e^{-\alpha_2})(1-e^{-\alpha_2})^{-1} \\ &= e^{\Lambda^1}(1-e^{-\alpha_1})^{-1}(1-e^{-\alpha_2})^{-1} \\ &\quad +\; e^{r_2\Lambda^1}(1-e^{-r_2\alpha_1})^{-1}(1-e^{-r_2\alpha_2})^{-1} \ . \end{aligned} \tag{4.8}$$

By Weyl invariance, similar formulas work at the other vertices, and so (4.4) agrees with $E[Pt_\lambda;Q]$.

We will proceed with our study of the expansion (4.5), postponing to [8] a proof that for all $\lambda \in P_\ge$, $E[Pt_\lambda;\lambda+Q] = B_\lambda$. Let us emphasize, however, that we are only relying on (4.4) and (4.5). Even if the sums of (4.3) and (4.4) are not identical, the expansion we are studying seems a natural one, and so should still be of value.

Consider the simplest nontrivial case: $X_r = A_2$. Then $S = \{\alpha_1, \alpha_2\}$, and most importantly, $R_> \backslash S = \{\alpha_1 + \alpha_2\} = \{\theta\}$. For A_2 then, we can write (4.4) as

$$\begin{aligned} \mathrm{ch}_\lambda \ &= \sum_{w\in W} e^{w\lambda} \left[\prod_{\alpha\in S} (1-e^{-w\alpha})^{-1} \right] (1-e^{-w\theta})^{-1} \\ &= B_\lambda + \sum_{w\in W} e^{w(\lambda-\theta)} \left[\prod_{\alpha\in S} (1-e^{-w\alpha})^{-1} \right] (1-e^{-w\theta})^{-1} \ , \end{aligned} \tag{4.9}$$

so that

$$\mathrm{ch}_\lambda \ = \ B_\lambda \ + \ \mathrm{ch}_{\lambda-\theta} \ . \tag{4.10}$$

It is also simple to show that if either $\nu_1 = 0$ or $\nu_2 = 0$, then $\mathrm{ch}_\nu = B_\nu$. Since $\theta = \Lambda^1 + \Lambda^2$ for A_2, we find that if $\lambda_{\min} := \min\{\lambda_1, \lambda_2\}$, then

$$\mathrm{ch}_\lambda \ = \ B_\lambda \ + \ B_{\lambda-\theta} \ + \ B_{\lambda-2\theta} \ + \ \ldots \ + \ B_{\lambda-\lambda_{\min}\theta} \ . \tag{4.11}$$

Eqn. (4.11) manifests the weight multiplicity pattern of A_2 representations.[2]

It is difficult to generalize the derivation just given of (4.11) to other algebras. (4.10) is easier, however. Consider $X_r = C_2$, with the short simple root labelled as α_1. Then

$$R_>\backslash S \ = \ \{\, 2\alpha_1 + \alpha_2 = 2\Lambda^1,\, \alpha_1 + \alpha_2 = \Lambda^2 \,\} \ . \tag{4.12}$$

Therefore

$$\begin{aligned} \mathrm{ch}_\lambda \ &= \sum_{w\in W} e^{w\lambda} \left[\prod_{\alpha\in S} (1-e^{-w\alpha})^{-1} \right] (1-e^{-w(2\Lambda^1)})^{-1} (1-e^{-w\Lambda^2})^{-1} \\ &= B_\lambda + \sum_{w\in W} e^{w\lambda} \left(e^{w(-2\Lambda^1)} + e^{w(-\Lambda^2)} - e^{w(-2\Lambda^1+\Lambda^2)} \right) \\ &\qquad\qquad\qquad \times \left[\prod_{\alpha\in R_>} (1-e^{-w\alpha})^{-1} \right] \ , \end{aligned} \tag{4.13}$$

so that

$$\mathrm{ch}_\lambda \ = \ B_\lambda \ + \ \mathrm{ch}_{\lambda-2\Lambda^1} \ + \ \mathrm{ch}_{\lambda-\Lambda^2} \ - \ \mathrm{ch}_{\lambda-2\Lambda^1-\Lambda^2} \ . \tag{4.14}$$

This recurrence relation is a bit more complicated than that of A_2, but can be analysed easily. We need the relation

$$\mathrm{ch}_\lambda \ = \ (\det w)\, \mathrm{ch}_{w.\lambda} \ , \tag{4.15}$$

derived from either (3.2) or (3.3). Using it with (4.14), we can establish

$$\mathrm{ch}_{\lambda_j\Lambda^j} \ = \ B_{\lambda_j\Lambda^j} \ + \ \mathrm{ch}_{(\lambda_j-2)\Lambda^j} \ , \qquad (j=1,2). \tag{4.16}$$

[2] This formula was derived in [2], where weight multiplicity patterns were studied for low-rank algebras, using methods close in spirit to the Kostant multiplicity formula [9]. The A_2 multiplicity pattern was known long before, however, by Wigner, for example; I thank Professor R. King for so informing me.

This immediately shows that $\mathrm{ch}_\lambda = B_\lambda$ for $\lambda \in \{0, \Lambda^1, \Lambda^2\}$, and that

$$\mathrm{ch}_{\lambda_j \Lambda^j} \;=\; B_{\lambda_j \Lambda^j} + B_{(\lambda_j - 2)\Lambda^j} + \ldots + B_{[\lambda_j]_2 \Lambda^j} \;, \quad (j = 1, 2). \tag{4.17}$$

Here $[\lambda_j]_2 := 0\,(1)$ if λ_j is even (odd). Now, if we define

$$v_\lambda \;:=\; \mathrm{ch}_\lambda \;-\; \mathrm{ch}_{\lambda - 2\Lambda^1} \;, \tag{4.18}$$

then the recursion relation (4.14) becomes

$$\begin{aligned} v_\lambda \;&= B_\lambda \;+\; v_{\lambda - \Lambda^2} \\ &= B_\lambda \;+\; B_{\lambda - \Lambda^2} \;+\; \ldots \;+\; B_{\lambda_1 \Lambda^1} \;, \end{aligned} \tag{4.19}$$

since (4.17) implies $v_{\lambda_1 \Lambda^1} = B_{\lambda_1 \Lambda^1}$. Solving (4.18) yields

$$\mathrm{ch}_\lambda \;=\; v_\lambda + v_{\lambda - 2\Lambda^1} + \ldots + v_{\Lambda^1 + \lambda_2 \Lambda^2} \tag{4.20}$$

for λ_1 odd; and

$$\mathrm{ch}_\lambda \;=\; v_\lambda + v_{\lambda - 2\Lambda^1} + \ldots + v_{2\Lambda^1 + \lambda_2 \Lambda^2} + \mathrm{ch}_{\lambda_2 \Lambda^2} \tag{4.21}$$

for λ_1 even. Using first (4.20), then (4.19), we find

$$\begin{aligned} \mathrm{ch}_\lambda \;&= B_\lambda + B_{\lambda - \Lambda^2} + \ldots + B_{\lambda_1 \Lambda^1} \\ &\quad + B_{\lambda - 2\Lambda^1} + B_{\lambda - 2\Lambda^1 - \Lambda^2} + \ldots + B_{(\lambda_1 - 2)\Lambda^1} \\ &\quad + \ldots \\ &\quad + B_{\Lambda^1 + \lambda_2 \Lambda^2} + B_{\Lambda^1 + (\lambda_2 - 1)\Lambda} + \ldots + B_{\Lambda^1} \end{aligned} \tag{4.22}$$

for λ_1 odd. Replacing (4.20) by (4.21) and (4.17), we get instead

$$\begin{aligned} \mathrm{ch}_\lambda \;&= B_\lambda + B_{\lambda - \Lambda^2} + \ldots + B_{\lambda_1 \Lambda^1} \\ &\quad + B_{\lambda - 2\Lambda^1} + B_{\lambda - 2\Lambda^1 - \Lambda^2} + \ldots + B_{(\lambda_1 - 2)\Lambda^1} \\ &\quad + \ldots \\ &\quad + B_{\lambda_2 \Lambda^2} + B_{(\lambda_2 - 2)\Lambda^2} + \ldots + B_{[\Lambda_2]_2 \Lambda^2} \end{aligned} \tag{4.23}$$

for λ_1 even. These C_2 results confirm those of [2].

Re-writing (4.10) as $B_\lambda = \mathrm{ch}_\lambda - \mathrm{ch}_{\lambda - \theta}$ gives us a hint as to how to generalize the method of computation of the $A_{\lambda,\mu}$ to all algebras. Expanding

$$B_\lambda \;=\; \sum_{\mu \le \lambda} A^{-1}_{\lambda,\mu}\, \mathrm{ch}_\mu \;, \tag{4.24}$$

is straightforward. Then finding $A_{\lambda,\mu}$ by diagonalizing the triangular matrix $(A^{-1}_{\lambda,\mu})$ is relatively easy.[3]

First, re-write (4.4) as

$$B_\lambda \;=\; \sum_{w \in W} e^{w\lambda} \left[\prod_{\gamma \in R_> \setminus S} (1 - e^{-w\gamma}) \right] \left[\prod_{\alpha \in R_>} (1 - e^{-w\alpha})^{-1} \right] \;. \tag{4.25}$$

Comparing to (3.5), we can therefore write

$$B_\lambda \;=\; \widehat{\mathrm{ch}}\; e^\lambda \prod_{\gamma \in R_> \setminus S} (1 - e^{-\gamma}) \;, \tag{4.26}$$

[3] The author learned this trick from a work of Professors J. Patera and R. T. Sharp [12], and has also used it elsewhere [7].

where we have defined

$$\widehat{\mathrm{ch}}\, e^{\lambda} \ := \ \mathrm{ch}_{\lambda}\ . \tag{4.27}$$

Of course, some of the terms of the expansion just written may be of the form ch_λ, but with $\lambda \notin P_{\geq}$. To find the coefficients $A^{-1}_{\lambda,\mu}$ with $\lambda, \mu \in P_{\geq}$, therefore, it is necessary to use the relation (4.15).

To write an explicit formula for $A^{-1}_{\lambda,\mu}$, we first define a partition function F by

$$\prod_{\gamma\in R_>\backslash S} (1-e^{-\gamma}) \ =: \sum_{\beta\in\mathbb{Z}_{\geq 0}R_>} F(\beta)\, e^{\beta}\ . \tag{4.28}$$

Then

$$B_\lambda \ = \sum_{\beta\in\mathbb{Z}_{\geq 0}R_>} F(\beta)\, \mathrm{ch}_{\lambda-\beta}\ . \tag{4.29}$$

Using (4.24) and (4.15), we find

$$A^{-1}_{\lambda,\mu} \ = \ \sum_{w\in W} (\det w)\, F(\lambda - w.\mu)\ . \tag{4.30}$$

For $X_r = G_2$, we have

$$\begin{aligned} R_>\backslash S \ &= \{\, \alpha_1+\alpha_2,\, 2\alpha_1+3\alpha_2,\, \alpha_1+2\alpha_2,\, \alpha_1+3\alpha_2 \,\}\ ,\\ &= \{\, \Lambda^1-\Lambda^2,\, \Lambda^1,\, \Lambda^2,\, -\Lambda^1+3\Lambda^2 \,\}\ . \end{aligned} \tag{4.31}$$

Therefore (4.26) yields

$$\begin{aligned} B_\lambda \ &= \mathrm{ch}_\lambda - \mathrm{ch}_{\lambda+\Lambda^1-3\Lambda^2} - \mathrm{ch}_{\lambda-\Lambda^2} + \mathrm{ch}_{\lambda+\Lambda^1-4\Lambda^2}\\ &\quad + \mathrm{ch}_{\lambda-\Lambda^1-\Lambda^2} - \mathrm{ch}_{\lambda-4\Lambda^2} - \mathrm{ch}_{\lambda-\Lambda^1+\Lambda^2}\\ &\quad + \mathrm{ch}_{\lambda-2\Lambda^2} + \mathrm{ch}_{\lambda-2\Lambda^1+\Lambda^2}\\ &\quad - \mathrm{ch}_{\lambda-\Lambda^1-2\Lambda^2} - \mathrm{ch}_{\lambda-2\Lambda^1} + \mathrm{ch}_{\lambda-\Lambda^1-3\Lambda^2}\ . \end{aligned} \tag{4.32}$$

A simple calculation then yields the matrix $(A_{\lambda,\mu})$:

$$\left[\begin{array}{cccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
2&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0\\
1&2&0&2&1&0&1&0&0&0&0&0&0&0&0&0\\
2&2&1&2&1&1&0&1&0&0&0&0&0&0&0&0\\
2&2&0&2&1&1&0&0&1&0&0&0&0&0&0&0\\
1&3&0&2&2&1&0&1&0&1&0&0&0&0&0&0\\
3&4&1&4&2&2&1&1&1&0&1&0&0&0&0&0\\
3&2&1&2&1&2&0&1&1&0&0&1&0&0&0&0\\
2&2&1&2&1&1&1&1&0&0&0&0&1&0&0&0\\
3&5&0&5&3&2&1&2&1&1&1&0&0&1&0&0\\
4&4&1&4&3&3&0&2&2&1&1&1&0&0&1&0\\
3&4&0&4&2&2&1&1&2&0&1&1&0&0&0&1
\end{array}\right] \tag{4.33}$$

Here all weights $\lambda =: (\lambda_1, \lambda_2)$ with $\lambda{\cdot}\theta = 2\lambda_1{+}\lambda_2 \leq 6$ are included, in the order

$$\begin{array}{l}(0,0),\ (0,1),\ (1,0),\ (0,2),\ (1,1),\ (0,3),\ (2,0),\ (1,2),\\ (0,4),\ (2,1),\ (1,3),\ (0,5),\ (3,0),\ (2,2),\ (1,4),\ (0,6)\ .\end{array} \tag{4.34}$$

The character polytope expansion (4.5) can be combined with the Weyl dimension formula for the dimension d_λ of the representation of highest weight λ:

$$d_\lambda \ = \ \prod_{\alpha\in R_>} \frac{(\lambda+\rho)\cdot\alpha}{\rho\cdot\alpha}\ . \tag{4.35}$$

The result is a formula for the number b_λ of lattice points counted by B_λ, that provides helpful checks on any expansions derived, since (4.24) and (4.5) imply $b_\lambda = \sum_\mu A^{-1}_{\lambda,\mu} d_\mu$ and $d_\mu = \sum_\sigma A_{\mu,\sigma} b_\sigma$, respectively. The formulas relevant to the simple rank-two algebras are

$$\begin{array}{lrcl}
A_2: & b_\lambda &=& (\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2 + 3\lambda_1 + 3\lambda_2 + 2)/2\ ,\\
C_2: & b_\lambda &=& \lambda_1^2 + 4\lambda_1\lambda_2 + 2\lambda_2^2 + 2\lambda_1 + 2\lambda_2 + 1\ ,\\
G_2: & b_\lambda &=& 9\lambda_1^2 + 12\lambda_1\lambda_2 + 3\lambda_2^2 + 3\lambda_1 + 3\lambda_2 + 1\ .
\end{array} \tag{4.36}$$

As our final example, consider $X_r = A_3$. The important subset of positive roots is

$$\begin{aligned} R_>\backslash S\ &= \{\,\alpha_{12},\, \alpha_{123},\, \alpha_{23}\,\}\\ &= \{\,\Lambda^1+\Lambda^2-\Lambda^3,\, \Lambda^1+\Lambda^3,\, -\Lambda^1+\Lambda^2+\Lambda^3\,\}\ , \end{aligned} \tag{4.37}$$

where $\alpha_{12} := \alpha_1 + \alpha_2$, etc. We therefore find

$$\begin{aligned} B_\lambda \ &= \mathrm{ch}_\lambda - \mathrm{ch}_{\lambda+\Lambda^1-\Lambda^2-\Lambda^3} - \mathrm{ch}_{\lambda-\Lambda^1-\Lambda^3} + \mathrm{ch}_{\lambda-\Lambda^2-2\Lambda^3} \\ &\quad - \mathrm{ch}_{\lambda-\Lambda^1-\Lambda^2+\Lambda^3} + \mathrm{ch}_{\lambda-2\Lambda^2} + \mathrm{ch}_{\lambda-2\Lambda^1-\Lambda^2} \\ &\quad - \mathrm{ch}_{\lambda-\Lambda^1-2\Lambda^2-\Lambda^3}\ . \end{aligned} \tag{4.38}$$

Using the Weyl dimension formula,

$$\begin{aligned} b_\lambda \ &= 1 + (11\lambda_1 + 14\lambda_2 + 11\lambda_3)/6 \\ &\quad + 4\lambda_1\lambda_2 + 3\lambda_1\lambda_3 + 4\lambda_2\lambda_3 + \lambda_1^2 + 2\lambda_2^2 + \lambda_3^2 \\ &\quad + (36\lambda_1\lambda_2\lambda_3 + 12\lambda_1\lambda_2^2 + 12\lambda_2^2\lambda_3 + 6\lambda_1^2\lambda_2 + 6\lambda_2\lambda_3^2 \\ &\quad + 9\lambda_1^2\lambda_3 + 9\lambda_1\lambda_3^2 + \lambda_1^3 + 4\lambda_2^3 + \lambda_3^3)/6 \end{aligned} \tag{4.39}$$

follows.

We present our results for all weights $\lambda =: (\lambda_1, \lambda_2, \lambda_3) \in P_\geq$ of A_3, with $\lambda \cdot \theta = \lambda_1 + \lambda_2 + \lambda_3 \leq 5$. The weights separate into four congruence classes, corresponding to the four shifted root lattices $\{0, \Lambda^1, \Lambda^2, \Lambda^3\} + Q$ that combine to form the weight lattice P.

For weights $\lambda, \mu \in Q$, we find the matrix $(A_{\lambda,\mu})$ is

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 2&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 2&1&1&1&1&0&0&0&1&0&0&0&0&0 \\ 3&2&0&0&0&0&0&0&0&1&0&0&0&0 \\ 2&2&0&1&0&1&0&0&0&1&1&0&0&0 \\ 2&2&0&0&1&0&0&1&0&1&0&1&0&0 \\ 1&1&0&1&1&0&0&0&1&0&0&0&1&0 \\ 1&1&0&1&1&0&0&0&1&0&0&0&0&1 \end{bmatrix} . \tag{4.40}$$

In the order used, the weights are

$$\begin{aligned} &(0,0,0),\ (1,0,1),\ (0,2,0),\ (2,1,0),\ (0,1,2),\ (4,0,0),\ (0,4,0), \\ &(0,0,4),\ (1,2,1),\ (2,0,2),\ (3,1,1),\ (1,1,3),\ (2,3,0),\ (0,3,2)\ . \end{aligned} \tag{4.41}$$

For weights $\lambda, \mu \in \Lambda^1 + Q$, we find $(A_{\lambda,\mu})$ to be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.42}$$

for weights $(\lambda_1, \lambda_2, \lambda_3)$ in the order

$$\begin{aligned} &(1,0,0),\ (0,1,1),\ (0,0,3),\ (1,2,0),\ (2,0,1),\ (0,3,1),\ (1,1,2), \\ &(3,1,0),\ (0,2,3),\ (1,0,4),\ (1,4,0),\ (2,2,1),\ (3,0,2),\ (5,0,0)\ . \end{aligned} \tag{4.43}$$

The results for weights in $\Lambda^3 + Q$ can be obtained from this, by replacing all weights $(\lambda_1, \lambda_2, \lambda_3)$ by their charge conjugates $(\lambda_3, \lambda_2, \lambda_1)$.

The matrix $(A_{\lambda,\mu})$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.44}$$

for weights $\lambda, \mu \in \Lambda^2 + Q$. The order of weights is

$$\begin{aligned} &(0,1,0),\ (0,0,2),\ (2,0,0),\ (0,3,0),\ (1,1,1),\ (1,0,3),\ (3,0,1),\\ &(0,2,2),\ (2,2,0),\ (0,1,4),\ (0,5,0),\ (1,3,1),\ (2,1,2),\ (4,1,0)\ . \end{aligned} \tag{4.45}$$

The preliminary calculations we have done for G_2 and A_3 confirm that the polytope-expansion multiplicities $A_{\lambda,\mu}$ can be computed easily by computer. Some patterns can already be seen from their results. To make some general statements, however, we plan to attempt an analysis of the recursion relations modeled on the one done above for C_2 [8]. Perhaps a relatively simple algorithm, of a combinatorial type, can be found for the calculation of the $A_{\lambda,\mu}$.

5. Conclusion

First, we'll summarize the main points. Then we'll discuss what still needs to be done, and what might be done.

We point out that the Brion formula applied to the weight polytopes Pt_λ produces a formula (4.4) that is remarkably similar to the Weyl character formula (3.5). The polytope expansion (4.5) of the Lie characters was therefore advocated. It makes manifest certain degeneracies in weight multiplicities beyond those explained by Weyl group symmetry, for example. The expansion multiplicities $A_{\lambda,\mu}$ were studied. A closed formula (4.30) was given for them, and methods for their computation were discussed and illustrated with the examples of A_2, C_2, G_2 and A_3.

Two conjectures need to be established. A general proof that the polytope-expansion coefficients are non-negative integers,

$$A_{\lambda,\mu} \ \in\ \mathbb{Z}_{\geq 0}\ , \tag{5.1}$$

would be helpful. It should also be determined if the exponential sum $E[Pt_\lambda; \lambda+Q]$ of the weight polytope Pt_λ is given by the Brion expression B_λ in (4.4), i.e., if

$$\sum_{\mu\in P(\lambda)} e^\mu \ =\ \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in S} (1-e^{-w\alpha})^{-1}\ , \tag{5.2}$$

for all integrable highest weights $\lambda \in P_\geq$. As mentioned above, however, we use only (4.4) and (4.5). Our results therefore have value even if this last equality is not always obeyed.

We'll now mention a few other possible applications of polytope theory to Lie theory.

This author was introduced to polytope theory in different contexts – in the study of tensor product multiplicities and the related affine fusion multiplicities (see [14] and references therein, and [4]), the fusion of Wess-Zumino-Witten conformal field theories. It would be interesting to consider applications of the Brion formula in those subjects.

For example, the Brion formula also allows the derivation of a character generating function relevant to affine fusion. Its derivation is a simple adaptation of that of the Patera-Sharp formula for the character generator of a simple Lie algebra [13]. We hope to report on it and its applications elsewhere.

As another example, consider the tensor-product multiplicity patterns for A_2 that were studied thoroughly long ago, in [11] and [10]. Not surprisingly, the pattern

is similar to the weight-multiplicity pattern of a single irreducible A_2 representation. Perhaps a polytope expansion in the spirit of (4.5) could manifest properties of the tensor-product patterns for all simple Lie algebras.

Recent work on tensor products has provided results that are valid for general classes of algebras, but on the simpler question of which weights appear in a given tensor product, ignoring their multiplicities. See [6] for a review. The Brion formula might be useful for this question, and perhaps also for the corresponding problem in affine fusion [1].

Finally, it is our hope that applications in the other direction will also be found, i.e., that certain techniques from Lie theory can also help in the study of polytopes. For example, the Brion formula has similarities with the Weyl character formula, as we have discussed. Are there polytope formulas that correspond to other formulas for Lie characters?

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